

THE METRIZABILITY OF THE GENERALIZED TANGENT BUNDLE OF DUAL OF A VECTOR BUNDLE

by
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Abstract

Two new classes of metrizable vector bundles have been presented in the papers [1] and [4]. The Lie algebroid generalized tangent bundle of a dual vector bundle is presented. This Lie algebroid is a new example of metrizable vector bundle. A new class of Hamilton spaces, called by use, generalized Hamilton (ρ, η) -space, Hamilton (ρ, η) -space and Cartan (ρ, η) -space are presented. The results obtained in the particular case of Lie algebroids emphasize the importance and the utility of our new method by work. In particular, if all morphisms are identities morphisms, then the classical results are obtained.

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1 Introduction

The study of the geometry of the usual Lie algebroid

$$((TT^*M, \tau_{T^*M}, T^*M), [\cdot, \cdot]_{TT^*M}, (Id_{TT^*M}, Id_{T^*M}))$$

with a metrical structure

$$g = g^{ij} dp_i \otimes dp_j \in \mathcal{T}_0^2(VTT^*M, \tau_{T^*M}, T^*M),$$

was extensively examined by geometers and physicists in the framework of generalized Hamilton space. (see [15]).

We know that a regular Hamiltonian on T^*M is a smooth function $T^*M \xrightarrow{H} \mathbb{R}$ such that the Hessian matrix with entries

$$g^{ij} = \frac{1}{2} \frac{\partial^2 H}{\partial p_i \partial p_j}$$

is everywhere nondegenerate. If the metrical structure of a generalized Hamilton space is determined by a regular Hamiltonian, then we obtain the Hamilton space. The concept of Hamilton, introduced by R. Miron in [14, 13], was intensively studied in [6, 7, 8, 10, 16, 17, ...] and it has been succesful as a geometric theory of the Hamiltonian fundamental function, the fundamental entity in Mechanics and Physics. In the general framework of generalized Lie algebroids, the geometry of the Hamilton fundamental function has been developed in the paper [3].

If H is square of a function on T^*M , positively, 1-homogeneous with respect to the momentum p_i , then an important class of Hamilton spaces, called by use Cartan spaces, were introduced by R. Miron [11, 12]. The geometry of Cartan space is a subgeometry of the geometry of the Lie algebroid

$$((TT^*M, \tau_{T^*M}, T^*M), [\cdot, \cdot]_{TT^*M}, (Id_{TT^*M}, Id_{T^*M})).$$

Important contributions to the geometry of Cartan spaces were obtained E. Cartan [5] and A. Kawaguchi [9], ...

The study of the metrizable in the general framework of generalized Lie algebroids was extensively studied in the papers [1, 4]. Using a generalized Lie algebroid, we obtain the Lie algebroid generalized tangent bundle

$$\left(\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right), [\cdot, \cdot]_{(\rho, \eta) T\overset{*}{E}}, \left(\overset{*}{\rho}, Id_{\overset{*}{E}} \right) \right)$$

of dual vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$. Using the basic notions and results presented in Sections 2, 3 and 4 we study the metrizable of this Lie algebroid in Section 5. In the particular case of Lie algebroids, we obtain important results. Moreover, we obtain new results for the metrizable of the usual Lie algebroid

$$\left(\left(T\overset{*}{E}, \tau_{\overset{*}{E}}, \overset{*}{E} \right), [\cdot, \cdot]_{T\overset{*}{E}}, \left(Id_{T\overset{*}{E}}, Id_{\overset{*}{E}} \right) \right).$$

Finally, in Section 6, we introduced a new class of Hamilton spaces, called by use *generalized Hamilton (ρ, η) -spaces, Hamilton (ρ, η) -spaces and Cartan (ρ, η) -spaces*.

In the particular case of Lie algebroids, new and important results are obtained. In particular, if $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$, then the classical results are obtained.

2 Preliminaries

Let **Vect**, **Liealg**, **Mod**, **Man** and $\mathbf{B}^{\mathbf{v}}$ be the category of real vector spaces, Lie algebras, modules, manifolds and vector bundles respectively.

We know that if $(E, \pi, M) \in |\mathbf{B}^{\mathbf{v}}|$, $\Gamma(E, \pi, M) = \{u \in \mathbf{Man}(M, E) : u \circ \pi = Id_M\}$ and $\mathcal{F}(M) = \mathbf{Man}(M, \mathbb{R})$, then $(\Gamma(E, \pi, M), +, \cdot)$ is a $\mathcal{F}(M)$ -module. If $(\varphi, \varphi_0) \in \mathbf{B}^{\mathbf{v}}((E, \pi, M), (E', \pi', M'))$ such that $\varphi_0 \in Iso_{\mathbf{Man}}(M, M')$, then, using the operation

$$\begin{array}{ccc} \mathcal{F}(M) \times \Gamma(E', \pi', M') & \xrightarrow{\quad \cdot \quad} & \Gamma(E', \pi', M') \\ (f, u') & \longmapsto & f \circ \varphi_0^{-1} \cdot u' \end{array}$$

it results that $(\Gamma(E', \pi', M'), +, \cdot)$ is a $\mathcal{F}(M)$ -module and we obtain the **Mod**-morphism

$$\begin{array}{ccc} \Gamma(E, \pi, M) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma(E', \pi', M') \\ u & \longmapsto & \Gamma(\varphi, \varphi_0)u \end{array}$$

defined by

$$\Gamma(\varphi, \varphi_0)u(y) = \varphi\left(u_{\varphi_0^{-1}(y)}\right),$$

for any $y \in M'$.

Let $M, N \in |\mathbf{Man}|$, $h \in Iso_{\mathbf{Man}}(M, N)$ and $\eta \in Iso_{\mathbf{Man}}(N, M)$.

We know (see [1, 3, 4]) that if $(F, \nu, N) \in |\mathbf{B}^{\mathbf{v}}|$ so that there exists

$$(\rho, \eta) \in \mathbf{B}^{\mathbf{v}}((F, \nu, N), (TM, \tau_M, M))$$

and also an operation

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[\cdot]_{F,h}} & \Gamma(F, \nu, N) \\ (u, v) & \longmapsto & [u, v]_{F,h} \end{array}$$

with the following properties:

*GLA*₁. the equality holds good

$$[u, f \cdot v]_{F,h} = f[u, v]_{F,h} + \Gamma(Th \circ \rho, h \circ \eta)(u) f \cdot v,$$

for all $u, v \in \Gamma(F, \nu, N)$ and $f \in \mathcal{F}(N)$.

*GLA*₂. the 4-tuple $(\Gamma(F, \nu, N), +, \cdot, [\cdot]_{F,h})$ is a Lie $\mathcal{F}(N)$ -algebra,

*GLA*₃. the **Mod**-morphism $\Gamma(Th \circ \rho, h \circ \eta)$ is a **LieAlg**-morphism of

$$\left(\Gamma(F, \nu, N), +, \cdot, [\cdot]_{F,h}\right)$$

source and

$$(\Gamma(TN, \tau_N, N), +, \cdot, [\cdot]_{TN})$$

target, then the triple $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$ is called *generalized Lie algebroid*.

In particular, if $h = Id_M = \eta$, then we obtain the definition of the Lie algebroid.

Let $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$ be an generalized Lie algebroid.

- Locally, for any $\alpha, \beta \in \overline{1, p}$, we set $[t_\alpha, t_\beta]_{F, h} = L_{\alpha\beta}^\gamma t_\gamma$. We easily obtain that $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$, for any $\alpha, \beta, \gamma \in \overline{1, p}$.

The real local functions $L_{\alpha\beta}^\gamma$, $\alpha, \beta, \gamma \in \overline{1, p}$ will be called the *structure functions of the generalized Lie algebroid* $\left((F, \nu, N), [\cdot, \cdot]_{F, h}, (\rho, \eta)\right)$.

- We assume the following diagrams:

$$\begin{array}{ccccc}
F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN \\
\downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\
N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \\
(\chi^{\tilde{i}}, z^\alpha) & & (x^i, y^i) & & (\chi^{\tilde{i}}, z^{\tilde{i}})
\end{array}$$

where $i, \tilde{i} \in \overline{1, m}$ and $\alpha \in \overline{1, p}$.

If

$$\begin{aligned}
(\chi^{\tilde{i}}, z^\alpha) &\longrightarrow (\chi^{\tilde{i}'}(\chi^{\tilde{i}}), z^{\alpha'}(\chi^{\tilde{i}}, z^\alpha)), \\
(x^i, y^i) &\longrightarrow (x^{\tilde{i}'}(x^i), y^{\tilde{i}'}(x^i, y^i))
\end{aligned}$$

and

$$(\chi^{\tilde{i}}, z^{\tilde{i}}) \longrightarrow (\chi^{\tilde{i}'}(\chi^{\tilde{i}}), z^{\tilde{i}'}(\chi^{\tilde{i}}, z^{\tilde{i}})),$$

then

$$z^{\alpha'} = \Lambda_\alpha^{\alpha'} z^\alpha,$$

$$y^{\tilde{i}'} = \frac{\partial x^{\tilde{i}'}}{\partial x^i} y^i$$

and

$$z^{\tilde{i}'} = \frac{\partial \chi^{\tilde{i}'}}{\partial \chi^{\tilde{i}}} z^{\tilde{i}}.$$

- We assume that $(\theta, \mu) \stackrel{put}{=} (Th \circ \rho, h \circ \eta)$. If $z^\alpha t_\alpha \in \Gamma(F, \nu, N)$ is arbitrary, then

$$\begin{aligned}
(2.1) \quad & \Gamma(Th \circ \rho, h \circ \eta)(z^\alpha t_\alpha) f(h \circ \eta(\varkappa)) = \\
& = \left(\theta_\alpha^{\tilde{i}} z^\alpha \frac{\partial f}{\partial \varkappa^{\tilde{i}}} \right) (h \circ \eta(\varkappa)) = \left((\rho_\alpha^i \circ h)(z^\alpha \circ h) \frac{\partial f \circ h}{\partial x^i} \right) (\eta(\varkappa)),
\end{aligned}$$

for any $f \in \mathcal{F}(N)$ and $\varkappa \in N$.

The coefficients ρ_α^i respectively $\theta_\alpha^{\tilde{i}}$ change to $\rho_{\alpha'}^{\tilde{i}'}$ respectively $\theta_{\alpha'}^{\tilde{i}'}$ according to the rule:

$$(2.2) \quad \rho_{\alpha'}^{\tilde{i}'} = \Lambda_\alpha^{\alpha'} \rho_\alpha^i \frac{\partial x^{\tilde{i}'}}{\partial x^i},$$

respectively

$$(2.3) \quad \theta_{\alpha'}^{\tilde{i}'} = \Lambda_\alpha^{\alpha'} \theta_\alpha^{\tilde{i}} \frac{\partial \varkappa^{\tilde{i}'}}{\partial \varkappa^{\tilde{i}}},$$

where

$$\|\Lambda_{\alpha'}^{\alpha}\| = \|\Lambda_\alpha^{\alpha'}\|^{-1}.$$

Remark 2.1 The following equalities hold good:

$$(2.4) \quad \rho_\alpha^i \circ h \frac{\partial f \circ h}{\partial x^i} = \left(\theta_\alpha^i \frac{\partial f}{\partial x^i} \right) \circ h, \forall f \in \mathcal{F}(N).$$

and

$$(2.5) \quad \left(L_{\alpha\beta}^\gamma \circ h \right) \left(\rho_\gamma^k \circ h \right) = \left(\rho_\alpha^i \circ h \right) \frac{\partial \left(\rho_\beta^k \circ h \right)}{\partial x^i} - \left(\rho_\beta^j \circ h \right) \frac{\partial \left(\rho_\alpha^k \circ h \right)}{\partial x^j}.$$

If $(E, \pi, M) \in |\mathbf{B}^v|$, then we have the \mathbf{B}^v -morphism

$$(2.6) \quad \begin{array}{ccc} \pi^{**}(h^*F) & \hookrightarrow & F \\ \pi^{**}(h^*\nu) \downarrow & & \downarrow \nu \\ E & \xrightarrow{h \circ \pi} & N \end{array}$$

Let $\left(\pi^{**}(h^*F), Id_E \right)$ be the \mathbf{B}^v -morphism of $\left(\pi^{**}(h^*F), \pi^{**}(h^*\nu), E \right)$ source and $\left(TE, \tau_E^*, E \right)$ target, where

$$(2.7) \quad \begin{array}{ccc} \pi^{**}(h^*F) & \xrightarrow{\pi^{**}(h^*F)} & TE \\ Z^\alpha T_\alpha \left(u_x \right) & \longmapsto & \left(Z^\alpha \cdot \rho_\alpha^i \circ h \circ \pi^* \right) \frac{\partial}{\partial x^i} \left(u_x \right) \end{array}$$

Using the operation

$$\Gamma \left(\pi^{**}(h^*F), \pi^{**}(h^*\nu), E \right)^2 \xrightarrow{[\cdot]_{\pi^{**}(h^*F)}} \Gamma \left(\pi^{**}(h^*F), \pi^{**}(h^*\nu), E \right)$$

defined by

$$(2.8) \quad \begin{aligned} [T_\alpha, T_\beta]_{\pi^{**}(h^*F)} &= \left(L_{\alpha\beta}^\gamma \circ h \circ \pi^* \right) T_\gamma, \\ [T_\alpha, fT_\beta]_{\pi^{**}(h^*F)} &= f \left(L_{\alpha\beta}^\gamma \circ h \circ \pi^* \right) T_\gamma + \left(\rho_\alpha^i \circ h \circ \pi^* \right) \frac{\partial f}{\partial x^i} T_\beta, \\ [fT_\alpha, T_\beta]_{\pi^{**}(h^*F)} &= -[T_\beta, fT_\alpha]_{\pi^{**}(h^*F)}, \end{aligned}$$

for any $f \in \mathcal{F} \left(E \right)$, it results that

$$\left(\left(\pi^{**}(h^*F), \pi^{**}(h^*\nu), E \right), [\cdot]_{\pi^{**}(h^*F)}, \left(\pi^{**}(h^*F), Id_E^* \right) \right)$$

is a Lie algebroid.

3 Natural and adapted basis

In the following we consider the following diagram:

$$(3.1) \quad \begin{array}{ccc} \overset{*}{E} & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\ \overset{*}{\pi} \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E, \pi, M) \in |\mathbf{B}^{\mathbf{v}}|$ and $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))$ is a generalized Lie algebroid.

We take (x^i, p_a) as canonical local coordinates on $(\overset{*}{E}, \overset{*}{\pi}, M)$, where $i \in \overline{1, m}$ and $a \in \overline{1, r}$. Let

$$(x^i, p_a) \longrightarrow (x^{\check{i}}(x^i), p_{a'}(x^i, p_a))$$

be a change of coordinates on $(\overset{*}{E}, \overset{*}{\pi}, M)$. Then the coordinates p_a change to $p_{a'}$ by the rule:

$$(3.2) \quad p_{a'} = M_{a'}^a p_a.$$

Let

$$(3.3) \quad \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_a} \right) \stackrel{put}{=} \left(\overset{*}{\partial}_i, \overset{\cdot a}{\partial} \right)$$

be the natural base of the dual tangent Lie algebroid $\left((TE, \tau_E^*, E), [\cdot, \cdot]_{TE^*}, (Id_{TE^*}, Id_E^*) \right)$.

For any sections

$$Z^\alpha T_\alpha \in \Gamma \left(\overset{*}{\pi}^* (h^* F), \overset{*}{\pi}^* (h^* F), \overset{*}{E} \right)$$

and

$$Y_a \overset{\cdot a}{\partial} \in \Gamma \left(VTE, \tau_E^*, E \right)$$

we obtain the section

$$\begin{aligned} Z^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a \overset{\cdot a}{\tilde{\partial}} &=: Z^\alpha \left(T_\alpha \oplus \left(\rho_\alpha^i \circ h \circ \overset{*}{\pi} \right) \overset{*}{\partial}_i \right) + Y_a \left(0_{\overset{*}{\pi}^* (h^* F)} \oplus \overset{\cdot a}{\partial} \right) \\ &= Z^\alpha T_\alpha \oplus \left(Z^\alpha \left(\rho_\alpha^i \circ h \circ \overset{*}{\pi} \right) \overset{*}{\partial}_i + Y_a \overset{\cdot a}{\partial} \right) \in \Gamma \left(\overset{*}{\pi}^* (h^* F) \oplus TE, \overset{*}{\pi}^* \oplus, \overset{*}{E} \right). \end{aligned}$$

Since we have

$$\begin{aligned} Z^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a \overset{\cdot a}{\tilde{\partial}} &= 0 \\ \Updownarrow \\ Z^\alpha T_\alpha = 0 \wedge Z^\alpha \left(\rho_\alpha^i \circ h \circ \overset{*}{\pi} \right) \overset{*}{\partial}_i + Y_a \overset{\cdot a}{\partial} &= 0, \end{aligned}$$

it implies $Z^\alpha = 0$, $\alpha \in \overline{1, p}$ and $Y_a = 0$, $a \in \overline{1, r}$.

Therefore, the sections $\overset{*}{\tilde{\partial}}_1, \dots, \overset{*}{\tilde{\partial}}_p, \overset{\cdot 1}{\tilde{\partial}}, \dots, \overset{\cdot r}{\tilde{\partial}}$ are linearly independent.

We consider the vector subbundle $\left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)$ of the vector bundle $\left(\overset{*}{\pi}^* (h^* F) \oplus TE^*, \overset{\oplus}{\pi}, E^* \right)$, for which the $\mathcal{F} \left(E^* \right)$ -module of sections is the $\mathcal{F} \left(E^* \right)$ -submodule of $\left(\Gamma \left(\overset{*}{\pi}^* (h^* F) \oplus TE^*, \overset{\oplus}{\pi}, E^* \right), +, \cdot \right)$, generated by the set of sections $\left(\overset{*}{\tilde{\partial}}_\alpha, \overset{\cdot a}{\tilde{\partial}} \right)$ which is called the *natural* (ρ, η) -base.

The matrix of coordinate transformation on $\left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)$ at a change of fibred charts is

$$(3.4) \quad \left\| \begin{array}{cc} \Lambda_\alpha^{\alpha'} \circ h \circ \overset{*}{\pi} & 0 \\ \left(\rho_a^i \circ h \circ \overset{*}{\pi} \right) \frac{\partial M_b^{\alpha'} \circ \overset{*}{\pi}}{\partial x_i} y^b & M_a^{\alpha'} \circ \overset{*}{\pi} \end{array} \right\|.$$

We have the following

Theorem 3.1 Let $\left(\overset{*}{\tilde{\rho}}, Id_E^* \right)$ be the \mathbf{B}^v -morphism of $\left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)$ source and $\left(TE^*, \tau_E^*, E^* \right)$ target, where

$$(3.5) \quad \begin{array}{c} (\rho, \eta) TE^* \xrightarrow{\overset{*}{\tilde{\rho}}} TE^* \\ \left(Z^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a \overset{\cdot a}{\tilde{\partial}} \right) (\overset{*}{u}_x) \longmapsto \left(Z^\alpha \left(\rho_\alpha^i \circ h \circ \overset{*}{\pi} \right) \overset{*}{\partial}_i + Y_a \overset{\cdot a}{\partial} \right) (\overset{*}{u}_x) \end{array}$$

Using the operation

$$\Gamma \left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)^2 \xrightarrow{[\cdot]_{(\rho, \eta) TE^*}} \Gamma \left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)$$

defined by

$$(3.6) \quad \begin{aligned} & \left[\left(Z_1^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a^1 \overset{\cdot a}{\tilde{\partial}} \right), \left(Z_2^\beta \overset{*}{\tilde{\partial}}_\beta + Y_b^2 \overset{\cdot b}{\tilde{\partial}} \right) \right]_{(\rho, \eta) TE^*} \\ &= \left[Z_1^\alpha T_a, Z_2^\beta T_b \right]_{\overset{*}{\pi}^* (h^* F)} \oplus \left[\left(\rho_\alpha^i \circ h \circ \overset{*}{\pi} \right) Z_1^\alpha \overset{*}{\partial}_i + Y_a^1 \overset{\cdot a}{\partial}, \right. \\ & \quad \left. \left(\rho_\beta^j \circ h \circ \overset{*}{\pi} \right) Z_2^\beta \overset{*}{\partial}_j + Y_b^2 \overset{\cdot b}{\partial} \right]_{TE^*}, \end{aligned}$$

for any $\left(Z_1^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a^1 \overset{\cdot a}{\tilde{\partial}} \right)$ and $\left(Z_2^\beta \overset{*}{\tilde{\partial}}_\beta + Y_b^2 \overset{\cdot b}{\tilde{\partial}} \right)$, we obtain that the couple

$$\left([\cdot]_{(\rho, \eta) TE^*}, \left(\overset{*}{\tilde{\rho}}, Id_E^* \right) \right)$$

is a Lie algebroid structure for the vector bundle $\left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)$.

The Lie algebroid

$$\left(\left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right), [\cdot, \cdot]_{(\rho, \eta) TE^*}, \left(\tilde{\rho}, Id_E^* \right) \right),$$

is called the *Lie algebroid generalized tangent bundle of dual vector bundle* $\left(E^*, \pi^*, M \right)$.

We consider the \mathbf{B}^V -morphism $\left((\rho, \eta) \pi^!, Id_E^* \right)$ given by the commutative diagram

$$(3.7) \quad \begin{array}{ccc} (\rho, \eta) TE^* & \xrightarrow{(\rho, \eta) \pi^!} & \pi^* (h^* F) \\ (\rho, \eta) \tau_E^* \downarrow & & \downarrow pr_1 \\ E^* & \xrightarrow{id_E^*} & E^* \end{array}$$

This is defined as:

$$(3.8) \quad (\rho, \eta) \pi^! \left(\tilde{Z}^\alpha \tilde{\partial}_\alpha^* + Y_a \tilde{\partial}^{\cdot a} \right) \left(u_x^* \right) = \left(\tilde{Z}^\alpha \tilde{T}_\alpha \right) \left(u_x^* \right),$$

for any $\tilde{Z}^\alpha \tilde{\partial}_\alpha^* + Y_a \tilde{\partial}^{\cdot a} \in \left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)$.

Using the \mathbf{B}^V -morphisms (2.6) and (3.7) we obtain the *tangent* (ρ, η) -application $\left((\rho, \eta) T\pi^*, h \circ \pi^* \right)$ of $\left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)$ source and (F, ν, N) target.

Definition 3.1 The kernel of the tangent (ρ, η) -application is written

$$\left(V(\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)$$

and it is called *the vertical subbundle*.

We remark that the set $\left\{ \tilde{\partial}^{\cdot a}, a \in \overline{1, r} \right\}$ is a base of the $\mathcal{F}(E^*)$ -module

$$\left(\Gamma \left(V(\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right), +, \cdot \right).$$

Proposition 3.1 *The short sequence of vector bundles*

$$(3.9) \quad \begin{array}{ccccccccc} 0 & \xrightarrow{i} & V(\rho, \eta) TE^* & \xrightarrow{i} & (\rho, \eta) TE^* & \xrightarrow{(\rho, \eta) \pi^!} & \pi^* (h^* F) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E^* & \xrightarrow{Id_E^*} & E^* & \xrightarrow{Id_E^*} & E^* & \xrightarrow{Id_E^*} & E^* & \xrightarrow{Id_E^*} & E^* \end{array}$$

is exact.

Let $(\rho, \eta) \Gamma$ be a (ρ, η) -connection for the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$, i. e. a **Man**-morphism of $(\rho, \eta) T \overset{*}{E}$ source and $V(\rho, \eta) T \overset{*}{E}$ target defined by

$$(3.10) \quad (\rho, \eta) \Gamma \left(\tilde{Z}^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_b \overset{\cdot}{\tilde{\partial}}^b \right) \left(\overset{*}{u}_x \right) = \left(Y_b - (\rho, \eta) \Gamma_{b\alpha} \tilde{Z}^\alpha \right) \overset{\cdot}{\tilde{\partial}}^b \left(\overset{*}{u}_x \right),$$

such that the **B^v**-morphism $\left((\rho, \eta) \Gamma, Id_{\overset{*}{E}} \right)$ is a split to the left in the previous exact sequence. Its components satisfy the law of transformation

$$(3.11) \quad (\rho, \eta) \Gamma_{b\gamma'} = M_b^{b' \circ \pi^*} \left[- \left(\rho_\gamma^i \circ h \circ \pi^* \right) \frac{\partial M_{b'}^{\alpha' \circ \pi^*}}{\partial x^i} p_{\alpha'} + (\rho, \eta) \Gamma_{b\gamma} \right] \left(\Lambda_{\gamma'}^\gamma \circ h \circ \pi^* \right).$$

In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain

$$(3.11)' \quad \rho \Gamma_{b\gamma} = M_b^{b' \circ \pi^*} \left[- \left(\rho_\gamma^i \circ \pi^* \right) \frac{\partial M_{b'}^{\alpha' \circ \pi^*}}{\partial x^i} p_{\alpha'} + \rho \Gamma_{b\gamma} \right] \left(\Lambda_{\gamma'}^\gamma \circ \pi^* \right).$$

In the classical case, $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$, we obtain

$$(3.11)'' \quad \Gamma_{bk} = M_b^{b' \circ \pi^*} \left[- \frac{\partial M_{b'}^{\alpha' \circ \pi^*}}{\partial x^i} p_{\alpha'} + \Gamma_{bk} \right] \left(\frac{\partial x^k}{\partial x^{k'}} \circ \pi^* \right).$$

The kernel of the **B^v**-morphism $\left((\rho, \eta) \Gamma, Id_{\overset{*}{E}} \right)$ is written $\left(H(\rho, \eta) T \overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ and is called the *horizontal vector subbundle*.

We remark that the horizontal and the vertical vector subbundles are interior differential systems (see [2]) of the Lie algebroid generalized tangent bundle

$$\left(\left((\rho, \eta) T \overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right), [\cdot, \cdot]_{(\rho, \eta) T \overset{*}{E}}, \left(\tilde{\rho}, Id_{\overset{*}{E}} \right) \right).$$

We put the problem of finding a base for the $\mathcal{F} \left(\overset{*}{E} \right)$ -module

$$\left(\Gamma \left(H(\rho, \eta) T \overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right), +, \cdot \right)$$

of the type

$$\tilde{\delta}_\alpha^* = Z_\alpha^\beta \tilde{\delta}_\beta^* + Y_{a\alpha} \overset{\cdot}{\tilde{\partial}}^a, \alpha \in \overline{1, r}$$

which satisfies the following conditions:

$$(3.12) \quad \begin{aligned} \Gamma \left((\rho, \eta) \overset{*}{\pi}!, Id_{\overset{*}{E}} \right) \left(\tilde{\delta}_\alpha^* \right) &= T_\alpha, \\ \Gamma \left((\rho, \eta) \Gamma, Id_{\overset{*}{E}} \right) \left(\tilde{\delta}_\alpha^* \right) &= 0. \end{aligned}$$

Then we obtain the sections

$$(3.13) \quad \tilde{\delta}_\alpha^* = \overset{*}{\tilde{\partial}}_\alpha + (\rho, \eta) \Gamma_{b\alpha} \overset{\cdot}{\tilde{\partial}}^b = T_\alpha \oplus \left(\left(\rho_\alpha^i \circ h \circ \pi^* \right) \overset{*}{\tilde{\partial}}_i + (\rho, \eta) \Gamma_{b\alpha} \overset{\cdot}{\tilde{\partial}}^b \right).$$

such that their law of change is a tensorial law under a change of vector fiber charts.

The base $\left(\begin{smallmatrix} * & \cdot a \\ \tilde{\delta}_\alpha & \tilde{\partial} \end{smallmatrix}\right)$ will be called the *adapted* (ρ, η) -base.

Remark 3.2 The following equality holds good

$$(3.14) \quad \Gamma \left(\begin{smallmatrix} * \\ \tilde{\rho}, Id_E^* \end{smallmatrix} \right) \left(\begin{smallmatrix} * \\ \tilde{\delta}_\alpha \end{smallmatrix} \right) = \left(\rho_\alpha^i \circ h \circ \pi \right)^* \partial_i + (\rho, \eta) \Gamma_{b\alpha} \dot{\partial}^b.$$

Moreover, if $(\rho, \eta) \Gamma$ is the (ρ, η) -connection associated to a connection Γ (see [1]), then we obtain

$$(3.15) \quad \Gamma \left(\begin{smallmatrix} * \\ \tilde{\rho}, Id_E^* \end{smallmatrix} \right) \left(\begin{smallmatrix} * \\ \tilde{\delta}_\alpha \end{smallmatrix} \right) = \left(\rho_\alpha^i \circ h \circ \pi \right)^* \delta_i,$$

where $\left(\begin{smallmatrix} * \\ \delta_i, \dot{\partial}^a \end{smallmatrix}\right)$ is the adapted base for the $\mathcal{F} \left(\begin{smallmatrix} * \\ E \end{smallmatrix} \right)$ -module $\left(\Gamma \left(\begin{smallmatrix} * \\ TE, \tau_E^*, E \end{smallmatrix} \right), +, \cdot \right)$.

Let $(d\tilde{z}^\alpha, d\tilde{p}_a)$ be the natural dual (ρ, η) -base of natural (ρ, η) -base $\left(\begin{smallmatrix} * & \cdot a \\ \partial_\alpha & \tilde{\partial} \end{smallmatrix}\right)$.

This is determined by the equations

$$\left\{ \begin{array}{ll} \left\langle d\tilde{z}^\alpha, \begin{smallmatrix} * \\ \tilde{\partial}_\beta \end{smallmatrix} \right\rangle = \delta_\beta^\alpha, & \left\langle d\tilde{z}^\alpha, \begin{smallmatrix} \cdot b \\ \tilde{\partial} \end{smallmatrix} \right\rangle = 0, \\ \left\langle d\tilde{p}_a, \begin{smallmatrix} * \\ \tilde{\partial}_\beta \end{smallmatrix} \right\rangle = 0, & \left\langle d\tilde{p}_a, \begin{smallmatrix} \cdot b \\ \tilde{\partial} \end{smallmatrix} \right\rangle = \delta_a^b. \end{array} \right.$$

We consider the problem of finding a base for the $\mathcal{F} \left(\begin{smallmatrix} * \\ E \end{smallmatrix} \right)$ -module

$$\left(\Gamma \left(\left(\begin{smallmatrix} * \\ V(\rho, \eta) TE \end{smallmatrix} \right)^*, \left(\begin{smallmatrix} * \\ (\rho, \eta) \tau_E^* \end{smallmatrix} \right)^*, E \right), +, \cdot \right)$$

of the type

$$\delta\tilde{p}_a = \theta_{a\alpha} d\tilde{z}^\alpha + \omega_a^b d\tilde{p}_b, \quad a \in \overline{1, r}$$

which satisfies the following conditions:

$$(3.16) \quad \left\langle \delta\tilde{p}_a, \begin{smallmatrix} \cdot b \\ \tilde{\partial} \end{smallmatrix} \right\rangle = \delta_a^b \wedge \left\langle \delta\tilde{p}_a, \begin{smallmatrix} * \\ \tilde{\delta}_\alpha \end{smallmatrix} \right\rangle = 0,$$

We obtain the sections

$$(3.17) \quad \delta\tilde{p}_a = -(\rho, \eta) \Gamma_{a\alpha} d\tilde{z}^\alpha + d\tilde{p}_a, \quad a \in \overline{1, r}.$$

such that their changing rule is tensorial under a change of vector fiber charts. The base $(d\tilde{z}^\alpha, \delta\tilde{p}_a)$ will be called the *adapted dual* (ρ, η) -base.

4 Tensor d -fields. Distinguished linear (ρ, η) -connections

We consider the following diagram:

$$\begin{array}{ccc} \begin{array}{c} * \\ E \\ \downarrow \pi \\ M \end{array} & \begin{array}{c} (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\ \downarrow \nu \\ N \end{array} \\ & \xrightarrow{h} \end{array}$$

where $(E, \pi, M) \in |\mathbf{B}^{\mathbf{V}}|$ and $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))$ is a generalized Lie algebroid.

Let $(\rho, \eta) \Gamma$ be a (ρ, η) -connection for the vector bundle $(\overset{*}{E}, \overset{*}{\pi}, M)$.

Let

$$\left(\mathcal{T}_{q,s}^{p,r} \left((\rho, \eta) TE^*, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right), +, \cdot \right)$$

be the $\mathcal{F}(\overset{*}{E})$ -module of tensor fields by $(\overset{p,r}{q,s})$ -type from the generalized tangent bundle

$$\left(H(\rho, \eta) TE^* \oplus V(\rho, \eta) TE^*, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right).$$

An arbitrarily tensor field T is written by the form:

$$T = T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1}^* \otimes \dots \otimes \tilde{\delta}_{\alpha_p}^* \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \overset{\cdot b_1}{\tilde{\partial}} \otimes \dots \otimes \overset{\cdot b_s}{\tilde{\partial}} \otimes \delta \tilde{p}_{a_1} \otimes \dots \otimes \delta \tilde{p}_{a_r}.$$

Let

$$\left(\mathcal{T} \left((\rho, \eta) TE^*, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right), +, \cdot, \otimes \right)$$

be the tensor fields algebra of generalized tangent bundle $\left((\rho, \eta) TE^*, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$.

If $T_1 \in \mathcal{T}_{q_1, s_1}^{p_1, r_1} \left((\rho, \eta) TE^*, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ and $T_2 \in \mathcal{T}_{q_2, s_2}^{p_2, r_2} \left((\rho, \eta) TE^*, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$, then the components of product tensor field $T_1 \otimes T_2$ are the products of local components of T_1 and T_2 .

Therefore, we obtain $T_1 \otimes T_2 \in \mathcal{T}_{q_1+q_2, s_1+s_2}^{p_1+p_2, r_1+r_2} \left((\rho, \eta) TE^*, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$.

Let $\mathcal{DT} \left((\rho, \eta) TE^*, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ be the family of tensor fields

$$T \in \mathcal{T} \left((\rho, \eta) TE^*, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$$

for which there exists

$$T_1 \in \mathcal{T}_{q,0}^{p,0} \left((\rho, \eta) TE^*, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$$

and

$$T_2 \in \mathcal{T}_{0,s}^{0,r} \left((\rho, \eta) TE^*, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$$

such that $T = T_1 + T_2$.

The $\mathcal{F}\left(\overset{*}{E}\right)$ -module $\left(\mathcal{DT}\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right), +, \cdot\right)$ will be called the *module of distinguished tensor fields* or the *module of tensor d-fields*.

Remark 4.1 The elements of

$$\Gamma\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

respectively

$$\Gamma(((\rho, \eta) T\overset{*}{E})^*, ((\rho, \eta) \tau_{\overset{*}{E}})^*, \overset{*}{E})$$

are tensor d -fields.

Definition 4.1 Let $(\rho, \eta) \Gamma$ be a (ρ, η) -connection for the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ and let

$$(4.1) \quad (X, T) \xrightarrow{(\rho, \eta) \overset{*}{D}} (\rho, \eta) \overset{*}{D}_X T$$

be a covariant (ρ, η) -derivative for the tensor algebra of generalized tangent bundle

$$\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

which preserves the horizontal and vertical IDS by parallelism. (see [2])

If (U, s_U) is a vector local $(m + r)$ -chart for $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, then the real local functions

$$\left((\rho, \eta) \overset{*}{H}_{\beta\gamma}^{\alpha}, (\rho, \eta) \overset{*}{H}_{b\gamma}^a, (\rho, \eta) \overset{*}{V}_{\beta}^{\alpha c}, (\rho, \eta) \overset{*}{V}_a^{bc}\right)$$

defined on $\pi^{*-1}(U)$ and determined by the following equalities:

$$(4.2) \quad \begin{aligned} (\rho, \eta) \overset{*}{D}_{\overset{*}{\delta}_{\gamma}}^{\alpha} \overset{*}{\delta}_{\beta} &= (\rho, \eta) \overset{*}{H}_{\beta\gamma}^{\alpha} \overset{*}{\delta}_{\alpha}, & (\rho, \eta) \overset{*}{D}_{\overset{*}{\delta}_{\gamma}}^{\overset{\cdot}{a}} \overset{*}{\delta} &= (\rho, \eta) \overset{*}{H}_{b\gamma}^a \overset{\cdot}{\delta}^b \\ (\rho, \eta) \overset{*}{D}_{\overset{*}{\delta}}^{\overset{\cdot}{c}} \overset{*}{\delta}_{\beta} &= (\rho, \eta) \overset{*}{V}_{\beta}^{\alpha c} \overset{*}{\delta}_{\alpha}, & (\rho, \eta) \overset{*}{D}_{\overset{*}{\delta}}^{\overset{\cdot}{b}} \overset{*}{\delta} &= (\rho, \eta) \overset{*}{V}_a^{bc} \overset{\cdot}{\delta}^a \end{aligned}$$

are the components of a linear (ρ, η) -connection

$$\left((\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V}\right)$$

for the generalized tangent bundle $\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$ which will be called the *distinguished linear (ρ, η) -connection*.

If $h = Id_M$, then the distinguished linear (Id_{TM}, Id_M) -connection will be called the *distinguished linear connection*.

The components of a distinguished linear connection $\left(\overset{*}{H}, \overset{*}{V}\right)$ will be denoted

$$\left(\overset{*}{H}_{jk}^i, \overset{*}{H}_{bk}^a, \overset{*}{V}_j^{ic}, \overset{*}{V}_a^{bc}\right).$$

Theorem 4.1 *If $\left((\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V}\right)$ is a distinguished linear (ρ, η) -connection for the generalized tangent bundle $\left((\rho, \eta) \overset{*}{TE}, (\rho, \eta) \overset{*}{\tau}_E, \overset{*}{E}\right)$, then its components satisfy the change relations:*

$$\begin{aligned}
(\rho, \eta) \overset{*}{H}_{\beta\gamma'} &= \Lambda_{\alpha'}^{\alpha'} \circ h \circ \pi \left[\Gamma \left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}} \right) \left(\overset{*}{\tilde{\delta}}_{\gamma} \right) \left(\Lambda_{\beta'}^{\alpha} \circ h \circ \pi \right) + \right. \\
&\quad \left. + (\rho, \eta) \overset{*}{H}_{\beta\gamma} \cdot \Lambda_{\beta'}^{\beta} \circ h \circ \pi \right] \cdot \Lambda_{\gamma'}^{\gamma} \circ h \circ \pi, \\
(\rho, \eta) \overset{*}{H}_{b\gamma'} &= M_a^{\alpha'} \circ \pi \left[\Gamma \left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}} \right) \left(\overset{*}{\tilde{\delta}}_{\gamma} \right) \left(M_b^a \circ \pi \right) + \right. \\
&\quad \left. + (\rho, \eta) \overset{*}{H}_{b\gamma} \cdot M_b^b \circ \pi \right] \cdot \Lambda_{\gamma'}^{\gamma} \circ h \circ \pi, \\
(\rho, \eta) \overset{*}{V}_{\beta'} &= \Lambda_{\alpha'}^{\alpha'} \circ h \circ \pi \cdot (\rho, \eta) \overset{*}{V}_{\beta} \cdot \Lambda_{\beta'}^{\beta} \circ h \circ \pi \cdot M_c^{\epsilon'} \circ \pi, \\
(\rho, \eta) \overset{*}{V}_{b'} &= M_a^{\alpha'} \circ \pi \cdot (\rho, \eta) \overset{*}{V}_b \cdot M_b^b \circ \pi \cdot M_c^{\epsilon'} \circ \pi.
\end{aligned}
\tag{4.3}$$

Corollary 4.1 *In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain*

$$\begin{aligned}
\rho \overset{*}{H}_{\beta\gamma'} &= \Lambda_{\alpha'}^{\alpha'} \circ \pi \left[\Gamma \left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}} \right) \left(\overset{*}{\tilde{\delta}}_{\gamma} \right) \left(\Lambda_{\beta'}^{\alpha} \circ \pi \right) + \rho \overset{*}{H}_{\beta\gamma} \cdot \Lambda_{\beta'}^{\beta} \circ \pi \right] \cdot \Lambda_{\gamma'}^{\gamma} \circ \pi, \\
\rho \overset{*}{H}_{b\gamma'} &= M_a^{\alpha'} \circ \pi \left[\Gamma \left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}} \right) \left(\overset{*}{\tilde{\delta}}_{\gamma} \right) \left(M_b^a \circ \pi \right) + \rho \overset{*}{H}_{b\gamma} \cdot M_b^b \circ \pi \right] \cdot \Lambda_{\gamma'}^{\gamma} \circ \pi, \\
\rho \overset{*}{V}_{\beta'} &= \Lambda_{\alpha'}^{\alpha'} \circ \pi \cdot \rho \overset{*}{V}_{\beta} \cdot \Lambda_{\beta'}^{\beta} \circ \pi \cdot M_c^{\epsilon'} \circ \pi, \\
\rho \overset{*}{V}_{b'} &= M_a^{\alpha'} \circ \pi \cdot \rho \overset{*}{V}_b \cdot M_b^b \circ \pi \cdot M_c^{\epsilon'} \circ \pi.
\end{aligned}
\tag{4.3}'$$

In the classical case, $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$, we obtain that the components of a distinguished linear connection (H, V) verify the change relations:

$$\begin{aligned}
\overset{*}{H}_{jk'} &= \frac{\partial x^{i'}}{\partial x^j} \circ \pi \cdot \left[\frac{\overset{*}{\delta}}{\delta x^k} \left(\frac{\partial x^i}{\partial x^j} \circ \pi \right) + \overset{*}{H}_{jk} \cdot \frac{\partial x^j}{\partial x^j} \circ \pi \right] \cdot \frac{\partial x^k}{\partial x^k} \circ \pi, \\
\overset{*}{H}_{bk'} &= M_a^{\alpha'} \circ \pi \cdot \left[\frac{\overset{*}{\delta}}{\delta x^k} \left(M_b^a \circ \pi \right) + \overset{*}{H}_{bk} \cdot M_b^b \circ \pi \right] \cdot \frac{\partial x^k}{\partial x^k} \circ \pi, \\
\overset{*}{V}_{j'} &= \frac{\partial x^{i'}}{\partial x^j} \circ \pi \cdot \overset{*}{V}_j \frac{\partial x^j}{\partial x^j} \circ \pi \cdot M_c^{\epsilon'} \circ \pi, \\
\overset{*}{V}_{b'} &= M_a^{\alpha'} \circ \pi \cdot \overset{*}{V}_b \cdot M_b^b \circ \pi \cdot M_c^{\epsilon'} \circ \pi.
\end{aligned}
\tag{4.3}''$$

Example 4.1 *If $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ is endowed with the (ρ, η) -connection $(\rho, \eta) \Gamma$, then the local real functions*

$$\left(\frac{\partial(\rho, \eta) \Gamma_{b\gamma}}{\partial p_a}, \frac{\partial(\rho, \eta) \Gamma_{b\gamma}}{\partial p_a}, 0, 0 \right)
\tag{4.4}$$

are the components of a distinguished linear (ρ, η) -connection for the generalized tangent bundle

$$\left((\rho, \eta) T^*E, (\rho, \eta) \tau_E^*, E \right),$$

which will be called the *Berwald linear (ρ, η) -connection*.

Theorem 4.2 *If the generalized tangent bundle $\left((\rho, \eta) T^*E, (\rho, \eta) \tau_E^*, E \right)$ is endowed with a distinguished linear (ρ, η) -connection $((\rho, \eta) \tilde{H}, (\rho, \eta) \tilde{V})$, then, for any*

$$X = Z^\gamma \tilde{\delta}_\gamma^* + Y_a \tilde{\partial}^{\cdot a} \in \Gamma \left((\rho, \eta) T^*E, (\rho, \eta) \tau_E^*, E \right)$$

and for any

$$T \in \mathcal{T}_{qs}^{pr} \left((\rho, \eta) T^*E, (\rho, \eta) \tau_E^*, E \right),$$

we obtain the formula:

$$\begin{aligned} & (\rho, \eta) D_X \left(T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1}^* \otimes \dots \otimes \tilde{\delta}_{\alpha_p}^* \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes \right. \\ & \quad \left. \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}^{\cdot b_1} \otimes \dots \otimes \tilde{\partial}^{\cdot b_s} \otimes \delta \tilde{p}_{a_1} \otimes \dots \otimes \delta \tilde{p}_{a_r} \right) = \\ (4.5) \quad & = Z^\gamma T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1}^* \otimes \dots \otimes \tilde{\delta}_{\alpha_p}^* \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}^{\cdot b_1} \otimes \dots \otimes \\ & \quad \otimes \tilde{\partial}^{\cdot b_s} \otimes \delta \tilde{p}_{a_1} \otimes \dots \otimes \delta \tilde{p}_{a_r} + Y_c T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |^c \tilde{\delta}_{\alpha_1}^* \otimes \dots \otimes \\ & \quad \otimes \tilde{\delta}_{\alpha_p}^* \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}^{\cdot b_1} \otimes \dots \otimes \tilde{\partial}^{\cdot b_s} \otimes \delta \tilde{p}_{a_1} \otimes \dots \otimes \delta \tilde{p}_{a_r}, \end{aligned}$$

where

$$\begin{aligned} & T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} = \Gamma \left(\tilde{\rho}, Id_E^* \right) \left(\tilde{\delta}_\gamma^* \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\ & \quad + (\rho, \eta) H_{\alpha \gamma}^{\alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) H_{\alpha \gamma}^{\alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} a_1 \dots a_r} \\ (4.6) \quad & - (\rho, \eta) H_{\beta_1 \gamma}^{\beta} T_{\beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) H_{\beta_q \gamma}^{\beta} T_{\beta_1 \dots \beta_{q-1} b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\ & - (\rho, \eta) H_{a \gamma}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} - \dots - (\rho, \eta) H_{a \gamma}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\ & + (\rho, \eta) H_{b_1 \gamma}^b T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) H_{b_s \gamma}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \end{aligned}$$

and

$$\begin{aligned}
(4.7) \quad & T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |^c = \Gamma \left(\tilde{\rho}, Id_E^* \right) \left(\tilde{\partial}^{\cdot c} \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
& + (\rho, \eta) V_{\alpha}^{* \alpha_1 c} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha \alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) V_{\alpha}^{* \alpha_p c} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} \alpha a_1 \dots a_r} \\
& - (\rho, \eta) V_{\beta_1}^{* \beta c} T_{\beta \beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) V_{\beta_q}^{* \beta c} T_{\beta_1 \dots \beta_{q-1} \beta b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
& - (\rho, \eta) V_a^{* a_1 c} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a a_2 \dots a_r} - \dots - (\rho, \eta) V_a^{* a_r c} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\
& + (\rho, \eta) V_{b_1}^{* bc} T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \dots + (\rho, \eta) V_{b_s}^{* bc} T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}.
\end{aligned}$$

Corollary 4.2 *In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain*

$$\begin{aligned}
(4.6)' \quad & T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} = \Gamma \left(\tilde{\rho}, Id_E^* \right) \left(\tilde{\delta}_{\gamma}^* \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
& + \rho H_{\alpha \gamma}^{* \alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha \alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + \rho H_{\alpha \gamma}^{* \alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} \alpha a_1 \dots a_r} \\
& - \rho H_{\beta_1 \gamma}^{* \beta} T_{\beta \beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - \rho H_{\beta_q \gamma}^{* \beta} T_{\beta_1 \dots \beta_{q-1} \beta b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
& - \rho H_{a \gamma}^{* a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a a_2 \dots a_r} - \dots - \rho H_{a \gamma}^{* a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\
& + \rho H_{b_1 \gamma}^{* b} T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \dots + \rho H_{b_s \gamma}^{* b} T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}
\end{aligned}$$

and

$$\begin{aligned}
(4.7)' \quad & T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |^c = \Gamma \left(\tilde{\rho}, Id_E^* \right) \left(\tilde{\partial}^{\cdot c} \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
& + \rho V_{\alpha}^{* \alpha_1 c} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha \alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + \rho V_{\alpha}^{* \alpha_p c} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} \alpha a_1 \dots a_r} \\
& - \rho V_{\beta_1}^{* \beta c} T_{\beta \beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - \rho V_{\beta_q}^{* \beta c} T_{\beta_1 \dots \beta_{q-1} \beta b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
& - \rho V_a^{* a_1 c} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a a_2 \dots a_r} - \dots - \rho V_a^{* a_r c} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\
& + \rho V_{b_1}^{* bc} T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \dots + \rho V_{b_s}^{* bc} T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}.
\end{aligned}$$

In the classical case, $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$, we obtain

$$\begin{aligned}
(4.6)'' \quad & T_{j_1 \dots j_q b_1 \dots b_s | k}^{i_1 \dots i_p a_1 \dots a_r} = \delta_k^* \left(T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} \right) \\
& + H_{ik}^{* i_1} T_{j_1 \dots j_q b_1 \dots b_s}^{i i_2 \dots i_p a_1 \dots a_r} + \dots + H_{ik}^{* i_p} T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_{p-1} i a_1 \dots a_r} \\
& - H_{j_1 k}^{* j} T_{j j_2 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} - \dots - H_{j_q k}^{* j} T_{j_1 \dots j_{q-1} j b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
& - H_{ak}^{* a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a a_2 \dots a_r} - \dots - H_{ak}^{* a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\
& + H_{b_1 k}^{* b} T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \dots + H_{b_s k}^{* b} T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}
\end{aligned}$$

and

$$\begin{aligned}
(4.7)'' \quad & T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} \mid^c = \dot{\partial}^c \left(T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} \right) \\
& + V_i^{* i_1 c} T_{j_1 \dots j_q b_1 \dots b_s}^{i i_2 \dots i_p a_1 \dots a_r} + \dots + V_i^{* i_p c} T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_{p-1} i a_1 \dots a_r} \\
& - V_{j_1}^{* j_1 c} T_{j j_2 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} - \dots - V_{j_q}^{* j_q c} T_{j_1 \dots j_{q-1} j b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} \\
& - V_a^{* a_1 c} T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a a_2 \dots a_r} - \dots - V_a^{* a_r c} T_{j_1 \dots j_q b_1 \dots b_s}^{i_1 \dots i_p a_1 \dots a_{r-1} a} \\
& + V_{b_1}^{* b c} T_{j_1 \dots j_q b b_2 \dots b_s}^{i_1 \dots i_p a_1 \dots a_r} \dots + V_{b_s}^{* b c} T_{j_1 \dots j_q b_1 \dots b_{s-1} b}^{i_1 \dots i_p a_1 \dots a_r}.
\end{aligned}$$

Definition 4.2 We assume that $(E, \pi, M) = (F, \nu, N)$.

If $(\rho, \eta) \Gamma$ is a (ρ, η) -connection for the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$ and

$$\left((\rho, \eta) \overset{*}{H}_{bc}, (\rho, \eta) \overset{*}{\tilde{H}}_{bc}, (\rho, \eta) \overset{*}{V}_b, (\rho, \eta) \overset{*}{\tilde{V}}_b \right)$$

are the components of a distinguished linear (ρ, η) -connection for the generalized tangent bundle $\left((\rho, \eta) \overset{*}{TE}, (\rho, \eta) \overset{*}{\tau}_E, \overset{*}{E} \right)$ such that

$$(\rho, \eta) \overset{*}{H}_{bc} = (\rho, \eta) \overset{*}{\tilde{H}}_{bc} \text{ and } (\rho, \eta) \overset{*}{V}_b = (\rho, \eta) \overset{*}{\tilde{V}}_b,$$

then we will say that the generalized tangent bundle $\left((\rho, \eta) \overset{*}{TE}, (\rho, \eta) \overset{*}{\tau}_E, \overset{*}{E} \right)$ is endowed with a normal distinguished linear (ρ, η) -connection on components

$$\left((\rho, \eta) \overset{*}{H}_{bc}, (\rho, \eta) \overset{*}{V}_b \right).$$

In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, the components of a normal distinguished linear (ρ, Id_M) -connection $\left(\rho \overset{*}{H}, \rho \overset{*}{V} \right)$ will be denoted $\left(\rho \overset{*}{H}_{bc}, \rho \overset{*}{V}_b \right)$.

In the classical case, $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$, the components of a normal distinguished linear (Id_{TM}, Id_M) -connection $\left(\overset{*}{H}, \overset{*}{V} \right)$ will be denoted $\left(\overset{*}{H}_{jk}, \overset{*}{V}_j \right)$.

5 The (ρ, η) -(pseudo)metrizability

We consider the following diagram:

$$\begin{array}{ccc}
\overset{*}{E} & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\
\overset{*}{\pi} \downarrow & & \downarrow \nu \\
M & \xrightarrow{h} & N
\end{array}$$

where $(E, \pi, M) \in |\mathbf{B}^V|$ and $((F, \nu, M), [\cdot, \cdot]_{F,h}, (\rho, \eta))$ is a generalized Lie algebroid. Let $(\rho, \eta) \Gamma$ be a (ρ, η) -connection for the vector bundle $\left(E, \pi^*, M\right)$ and let $\left((\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V}\right)$ be a distinguished linear (ρ, η) -connection for the generalized tangent bundle

$$\left((\rho, \eta) T \overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right).$$

Definition 5.1 A tensor d -field

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b \in \mathcal{DT}_{20}^{02} \left((\rho, \eta) T \overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

will be called *pseudometrical structure* if its components are symmetric and the matrices $\left\|g_{\alpha\beta} \left(\overset{*}{u}_x\right)\right\|$ and $\left\|g_{ab} \left(\overset{*}{u}_x\right)\right\|$ are nondegenerate, for any point $\overset{*}{u}_x \in \overset{*}{E}$.

Moreover, if the matrices $\left\|g_{\alpha\beta} \left(\overset{*}{u}_x\right)\right\|$ and $\left\|g_{ab} \left(\overset{*}{u}_x\right)\right\|$ has constant signature, then the tensor d -field G will be called *metrical structure*.

Let

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

be a (pseudo)metrical structure. If $\alpha, \beta \in \overline{1, p}$ and $a, b \in \overline{1, r}$, then for any vector local $(m+r)$ -chart $\left(U, \overset{*}{s}_U\right)$ of $\left(\overset{*}{E}, \pi^*, M\right)$, we consider the real functions

$$\overset{*}{\pi}^{-1}(U) \xrightarrow{\tilde{g}^{\beta\alpha}} \mathbb{R}$$

and

$$\overset{*}{\pi}^{-1}(U) \xrightarrow{\tilde{g}_{ba}} \mathbb{R}$$

such that

$$\left\|\tilde{g}^{\beta\alpha} \left(\overset{*}{u}_x\right)\right\| = \left\|g_{\alpha\beta} \left(\overset{*}{u}_x\right)\right\|^{-1}$$

and

$$\left\|\tilde{g}_{ba} \left(\overset{*}{u}_x\right)\right\| = \left\|g^{ab} \left(\overset{*}{u}_x\right)\right\|^{-1},$$

for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U) \setminus \left\{\overset{*}{0}_x\right\}$.

Definition 5.2 If around each point $x \in M$ it exists a local vector $m+r$ -chart $\left(U, \overset{*}{s}_U\right)$ and a local m -chart (U, ξ_U) such that $g_{\alpha\beta} \circ \overset{*}{s}_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, p)$ and $g^{ab} \circ \overset{*}{s}_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, p)$ depends only on x , for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U)$, then we will say that the (pseudo)metrical structure

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

is a *Riemannian (pseudo)metrical structure*.

If only the condition is verified:

” $g_{\alpha\beta} \circ \overset{*}{s}_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, p)$ depends only on x , for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U)$ ” respectively ” $g^{ab} \circ \overset{*}{s}_U^{-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, p)$ depends only on x , for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U)$ ”, then

we will say that the (pseudo)metrical structure G is a Riemannian \mathcal{H} -(pseudo)metrical structure respectively a Riemannian \mathcal{V} -(pseudo)metrical structure.

Definition 5.3 If around each point $x \in M$ there exists a local vector $m + r$ -chart (U, s_U^*) and a local m -chart (U, ξ_U) such that $g_{\alpha\beta} \circ s_U^{*-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, p)$ and $g^{ab} \circ s_U^{*-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, p)$ depends only on p , for any $u_x^* \in \pi^{*-1}(U)$, then we will say that the (pseudo)metrical structure

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

is a Minkowski (pseudo)metrical structure.

If only the condition is verified:

" $g_{\alpha\beta} \circ s_U^{*-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, p)$ depends only on p , for any $u_x^* \in \pi^{*-1}(U)$ " respectively " $g^{ab} \circ s_U^{*-1} \circ (\xi_U^{-1} \times Id_{\mathbb{R}^m})(x, p)$ depends only on p , for any $u_x^* \in \pi^{*-1}(U)$ ", then we will say that the (pseudo)metrical structure G is a Minkowski \mathcal{H} -(pseudo)metrical structure respectively a Minkowski \mathcal{V} -(pseudo)metrical structure.

Definition 5.4 If there exists a (pseudo)metrical structure

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

and a distinguished linear (ρ, η) -connection

$$\left((\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V} \right)$$

such that

$$(5.1) \quad (\rho, \eta) \overset{*}{D}_X G = 0, \quad \forall X \in \Gamma \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right).$$

then the generalized tangent bundle $\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ will be called (ρ, η) -(pseudo)metrizable

Condition (5.1) is equivalent with the following equalities:

$$(5.2) \quad g_{\alpha\beta}|_\gamma = 0, \quad g^{ab}|_\gamma = 0, \quad g_{\alpha\beta}|^c = 0, \quad g^{ab}|^c = 0.$$

If $g_{\alpha\beta}|_\gamma = 0$ and $g^{ab}|_\gamma = 0$, then we will say that the vector bundle $\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ is \mathcal{H} -(ρ, η)-(pseudo)metrizable.

If $g_{\alpha\beta}|^c = 0$ and $g^{ab}|^c = 0$, then we will say that the vector bundle $\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ is \mathcal{V} -(ρ, η)-(pseudo)metrizable.

Theorem 5.1 If $\left((\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V} \right)$ is a distinguished linear (ρ, η) -connection for the generalized tangent bundle $\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ and $G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g^{ab} \delta\tilde{p}_a \otimes$

$\delta\tilde{p}_b$ is a (pseudo)metrical structure, then the following real local functions:

$$\begin{aligned}
(\rho, \eta) \overset{*}{H}_{\beta\gamma}^{\alpha} &= \frac{1}{2}\tilde{g}^{\alpha\varepsilon} \left(\Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\gamma} \right) g_{\varepsilon\beta} + \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\beta} \right) g_{\varepsilon\gamma} - \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\varepsilon} \right) g_{\beta\gamma} \right. \\
&\quad \left. + g_{\theta\varepsilon} L_{\gamma\beta}^{\theta} \circ h \circ \overset{*}{\pi} - g_{\beta\theta} L_{\gamma\varepsilon}^{\theta} \circ h \circ \overset{*}{\pi} - g_{\theta\gamma} L_{\beta\varepsilon}^{\theta} \circ h \circ \overset{*}{\pi} \right), \\
(5.3) \quad (\rho, \eta) \overset{*}{H}_{b\gamma}^a &= (\rho, \eta) \overset{*}{\dot{H}}_{b\gamma}^a + \frac{1}{2}\tilde{g}_{be} g^{ea} \underset{|\gamma}{0}, \\
(\rho, \eta) \overset{*}{V}_{\beta}^{\alpha c} &= (\rho, \eta) \overset{*}{\dot{V}}_{\beta}^{\alpha c} + \frac{1}{2}\tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta} \underset{|\beta}{0^c}, \\
(\rho, \eta) \overset{*}{V}_b^{ac} &= \frac{1}{2}\tilde{g}_{be} \left(\Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{\cdot}{\tilde{\partial}}^c \right) g^{ea} + \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{\cdot}{\tilde{\partial}}^a \right) g^{ec} - \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{\cdot}{\tilde{\partial}}^e \right) g^{ac} \right)
\end{aligned}$$

are components of a distinguished linear (ρ, η) -connection such that the generalized tangent bundle $\left((\rho, \eta) TE, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ becomes (ρ, η) -(pseudo)metrizable.

Corollary 5.1 In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, then we obtain

$$\begin{aligned}
\rho \overset{*}{H}_{\beta\gamma}^{\alpha} &= \frac{1}{2}\tilde{g}^{\alpha\varepsilon} \left(\Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\gamma} \right) g_{\varepsilon\beta} + \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\beta} \right) g_{\varepsilon\gamma} - \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\varepsilon} \right) g_{\beta\gamma} \right. \\
&\quad \left. + g_{\theta\varepsilon} L_{\gamma\beta}^{\theta} \circ h \circ \overset{*}{\pi} - g_{\beta\theta} L_{\gamma\varepsilon}^{\theta} \circ h \circ \overset{*}{\pi} - g_{\theta\gamma} L_{\beta\varepsilon}^{\theta} \circ h \circ \overset{*}{\pi} \right), \\
(5.3)' \quad \rho \overset{*}{H}_{b\gamma}^a &= \rho \overset{*}{\dot{H}}_{b\gamma}^a + \frac{1}{2}\tilde{g}_{be} g^{ea} \underset{|\gamma}{0}, \\
\rho \overset{*}{V}_{\beta}^{\alpha c} &= \rho \overset{*}{\dot{V}}_{\beta}^{\alpha c} + \frac{1}{2}\tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta} \underset{|\beta}{0^c}, \\
\rho \overset{*}{V}_b^{ac} &= \frac{1}{2}\tilde{g}_{be} \left(\Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{\cdot}{\tilde{\partial}}^c \right) g^{ea} + \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{\cdot}{\tilde{\partial}}^a \right) g^{ec} - \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{\cdot}{\tilde{\partial}}^e \right) g^{ac} \right)
\end{aligned}$$

In the classical case, $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$, then we obtain

$$\begin{aligned}
(5.3)'' \quad \overset{*}{H}_{jk}^i &= \frac{1}{2}\tilde{g}^{ih} \left(\overset{*}{\delta}_k g_{hj} + \overset{*}{\delta}_j g_{hk} - \overset{*}{\delta}_h g_{jk} \right) \\
\overset{*}{H}_{bk}^a &= \overset{*}{\dot{H}}_{bk}^a + \frac{1}{2}\tilde{g}_{be} g^{ea} \underset{|\k}{0}, \\
\overset{*}{V}_j^{ic} &= \overset{*}{\dot{V}}_j^{ic} + \frac{1}{2}\tilde{g}^{ih} g_{hj} \underset{|\beta}{0^c}, \\
\overset{*}{V}_b^{ac} &= \frac{1}{2}\tilde{g}_{be} \left(\overset{\cdot}{\partial}^c g^{ea} + \overset{\cdot}{\partial}^a g^{ec} - \overset{\cdot}{\partial}^e g^{ac} \right)
\end{aligned}$$

Theorem 5.2 If the distinguished linear (ρ, η) -connection $\left((\rho, \eta) \overset{0}{H}, (\rho, \eta) \overset{0}{V} \right)$ coincides with the Berwald linear (ρ, η) -connection in the previous theorem, then the local real

functions:

$$\begin{aligned}
(\rho, \eta) \overset{c}{\underset{*}{H}}_{\beta\gamma}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \left(\Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\gamma} \right) g_{\epsilon\beta} + \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\beta} \right) g_{\epsilon\gamma} \right. \\
&\quad \left. - \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\epsilon} \right) g_{\beta\gamma} + g_{\theta\epsilon} L_{\gamma\beta}^{\theta} \circ h \circ \overset{*}{\pi} - g_{\beta\theta} L_{\gamma\epsilon}^{\theta} \circ h \circ \overset{*}{\pi} - g_{\theta\gamma} L_{\beta\epsilon}^{\theta} \circ h \circ \overset{*}{\pi} \right), \\
(5.4) \quad (\rho, \eta) \overset{c}{\underset{*}{H}}_{b\gamma}^a &= \frac{\partial(\rho, \eta) \Gamma_{b\gamma}}{\partial p_a} + \frac{1}{2} \tilde{g}_{be} g^{ea} \underset{|\gamma}{0}, \\
(\rho, \eta) \overset{c}{\underset{*}{V}}_{\beta}^{\alpha c} &= \frac{1}{2} \tilde{g}_{\beta\epsilon} \frac{\partial g^{\epsilon\alpha}}{\partial p_c}, \\
(\rho, \eta) \overset{c}{\underset{*}{V}}_b^{ac} &= \frac{1}{2} \tilde{g}_{be} \left(\frac{\partial g^{ea}}{\partial p_c} + \frac{\partial g^{ec}}{\partial p_a} - \frac{\partial g^{ac}}{\partial p_e} \right)
\end{aligned}$$

are the components of a distinguished linear (ρ, η) -connection such that the generalized tangent bundle $\left((\rho, \eta) T E, (\rho, \eta) \tau_E^*, E \right)$ becomes (ρ, η) -(pseudo)metrizable.

Moreover, if the (pseudo)metrical structure G is \mathcal{H} - and \mathcal{V} -Riemannian, then the local real functions:

$$\begin{aligned}
(\rho, \eta) \overset{c}{\underset{*}{H}}_{\beta\gamma}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \left(\rho_{\gamma}^k \circ h \circ \overset{*}{\pi} \frac{\partial g_{\epsilon\beta}}{\partial x^k} + \rho_{\beta}^j \circ h \circ \overset{*}{\pi} \frac{\partial g_{\epsilon\gamma}}{\partial x^j} - \rho_{\epsilon}^e \circ h \circ \overset{*}{\pi} \frac{\partial g_{\beta\gamma}}{\partial x^e} + \right. \\
&\quad \left. + g_{\theta\epsilon} L_{\gamma\beta}^{\theta} \circ h \circ \overset{*}{\pi} - g_{\beta\theta} L_{\gamma\epsilon}^{\theta} \circ h \circ \overset{*}{\pi} - g_{\theta\gamma} L_{\beta\epsilon}^{\theta} \circ h \circ \overset{*}{\pi} \right), \\
(5.5) \quad (\rho, \eta) \overset{c}{\underset{*}{H}}_{b\gamma}^a &= \frac{\partial(\rho, \eta) \Gamma_{b\gamma}}{\partial p_a} + \frac{1}{2} \tilde{g}_{be} \left(\rho_{\gamma}^i \circ h \circ \overset{*}{\pi} \frac{\partial g^{ea}}{\partial x^i} - \frac{\partial(\rho, \eta) \Gamma_{d\gamma}}{\partial p_e} g^{da} - \frac{\partial(\rho, \eta) \Gamma_{d\gamma}}{\partial p_a} g^{ed} \right), \\
(\rho, \eta) \overset{c}{\underset{*}{V}}_{\beta}^{\alpha c} &= 0, \\
(\rho, \eta) \overset{c}{\underset{*}{V}}_b^{ac} &= 0.
\end{aligned}$$

are the components of a distinguished linear (ρ, η) -connection such that the generalized tangent bundle $\left((\rho, \eta) T E, (\rho, \eta) \tau_E^*, E \right)$ becomes (ρ, η) -(pseudo)metrizable.

Corollary 5.2 In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, then we obtain

$$\begin{aligned}
\rho \overset{c}{\underset{*}{H}}_{\beta\gamma}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\epsilon} \left(\Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\gamma} \right) g_{\epsilon\beta} + \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\beta} \right) g_{\epsilon\gamma} \right. \\
&\quad \left. - \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\epsilon} \right) g_{\beta\gamma} + g_{\theta\epsilon} L_{\gamma\beta}^{\theta} \circ \overset{*}{\pi} - g_{\beta\theta} L_{\gamma\epsilon}^{\theta} \circ \overset{*}{\pi} - g_{\theta\gamma} L_{\beta\epsilon}^{\theta} \circ \overset{*}{\pi} \right), \\
(5.4)' \quad \rho \overset{c}{\underset{*}{H}}_{b\gamma}^a &= \frac{\partial \rho \Gamma_{b\gamma}}{\partial p_a} + \frac{1}{2} \tilde{g}_{be} g^{ea} \underset{|\gamma}{0}, \\
\rho \overset{c}{\underset{*}{V}}_{\beta}^{\alpha c} &= \frac{1}{2} \tilde{g}_{\beta\epsilon} \frac{\partial g^{\epsilon\alpha}}{\partial p_c}, \\
\rho \overset{c}{\underset{*}{V}}_b^{ac} &= \frac{1}{2} \tilde{g}_{be} \left(\frac{\partial g^{ea}}{\partial p_c} + \frac{\partial g^{ec}}{\partial p_a} - \frac{\partial g^{ac}}{\partial p_e} \right)
\end{aligned}$$

If the (pseudo)metrical structure G is \mathcal{H} - and \mathcal{V} -Riemannian, then

$$\begin{aligned}
(5.5)' \quad \rho \overset{c}{\underset{*}{H}}_{\beta\gamma}^{\alpha} &= \frac{1}{2} \tilde{g}^{\alpha\varepsilon} \left(\Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\gamma} \right) g_{\varepsilon\beta} + \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\beta} \right) g_{\varepsilon\gamma} \right. \\
&\quad \left. - \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E \right) \left(\overset{*}{\tilde{\delta}}_{\varepsilon} \right) g_{\beta\gamma} + g_{\theta\varepsilon} L_{\gamma\beta}^{\theta} \circ \overset{*}{\pi} - g_{\beta\theta} L_{\gamma\varepsilon}^{\theta} \circ \overset{*}{\pi} - g_{\theta\gamma} L_{\beta\varepsilon}^{\theta} \circ \overset{*}{\pi} \right), \\
\rho \overset{c}{\underset{*}{H}}_{b\gamma}^a &= \frac{\partial \rho \Gamma_{b\gamma}}{\partial p_a} + \frac{1}{2} \tilde{g}_{be} g^{ea} \underset{|\gamma}{0}, \\
\rho \overset{c}{\underset{*}{V}}_{\beta}^{\alpha c} &= 0 \\
\rho \overset{c}{\underset{*}{V}}_b &= 0
\end{aligned}$$

In the classicale case, $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$, then we obtain

$$\begin{aligned}
(5.4)'' \quad \overset{c}{\underset{*}{H}}_{jk}^i &= \frac{1}{2} \tilde{g}^{ih} \left(\overset{*}{\delta}_k g_{hj} + \overset{*}{\delta}_j g_{hk} - \overset{*}{\delta}_h g_{jk} \right) \\
\overset{c}{\underset{*}{H}}_{bk}^a &= \frac{\partial \Gamma_{bk}}{\partial p_a} + \frac{1}{2} \tilde{g}_{be} g^{ea} \underset{|k}{0}, \\
\overset{c}{\underset{*}{V}}_j^{ic} &= \frac{1}{2} \tilde{g}_{jh} \frac{\partial g^{hi}}{\partial p_c}, \\
\overset{c}{\underset{*}{V}}_b^{ac} &= \frac{1}{2} \tilde{g}_{be} \left(\frac{\partial g^{ea}}{\partial p_c} + \frac{\partial g^{ec}}{\partial p_a} - \frac{\partial g^{ac}}{\partial p_e} \right)
\end{aligned}$$

If the (pseudo)metrical structure G is \mathcal{H} - and \mathcal{V} -Riemannian, then

$$\begin{aligned}
(5.5)'' \quad \overset{c}{\underset{*}{H}}_{jk}^i &= \frac{1}{2} \tilde{g}^{ih} \left(\frac{\partial g_{hj}}{\partial x^k} + \frac{\partial g_{hk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h} \right) \\
\overset{c}{\underset{*}{H}}_{bk}^a &= \frac{\partial \Gamma_{bk}}{\partial p_a} + \frac{1}{2} \tilde{g}_{be} \left(\frac{\partial g^{ea}}{\partial x^i} - \frac{\partial \Gamma_{dk}}{\partial p_e} g^{da} - \frac{\partial \Gamma_{dk}}{\partial p_a} g^{ed} \right), \\
\overset{c}{\underset{*}{V}}_j^{ic} &= 0, \quad \overset{c}{\underset{*}{V}}_b^{ac} = 0.
\end{aligned}$$

Theorem 5.3 Let $(\rho, \eta) \Gamma$ be a (ρ, η) -connection for the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$. Let

$$\left((\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V} \right)$$

be a distinguished linear (ρ, η) -connection for

$$\left((\rho, \eta) T \overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$$

and let

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

be a (pseudo)metrical structure. Let

$$(5.6) \quad \begin{aligned} O_{\beta\gamma}^{\alpha\varepsilon} &= \frac{1}{2} (\delta_\beta^\alpha \delta_\gamma^\varepsilon - g_{\beta\gamma} \tilde{g}^{\alpha\varepsilon}), & O_{\beta\gamma}^{*\alpha\varepsilon} &= \frac{1}{2} (\delta_\beta^\alpha \delta_\gamma^\varepsilon + g_{\beta\gamma} \tilde{g}^{\alpha\varepsilon}), \\ O_{bc}^{ae} &= \frac{1}{2} (\delta_b^a \delta_c^e - \tilde{g}_{bc} g^{ae}), & O_{bc}^{*ae} &= \frac{1}{2} (\delta_b^a \delta_c^e + \tilde{g}_{bc} g^{ae}), \end{aligned}$$

be the Obata operators.

If the real local functions $X_{\beta\gamma}^\alpha, X_\beta^{\alpha c}, Y_{b\gamma}^a, Y_b^{ac}$ are components of tensor fields, then the local real functions given in the following:

$$(5.7) \quad \begin{aligned} (\rho, \eta) \overset{*}{H}_{\beta\gamma}^{\alpha} &= (\rho, \eta) \overset{*}{H}_{\beta\gamma}^{\overset{c}{\alpha}} + O_{\gamma\eta}^{\alpha\varepsilon} X_{\varepsilon\beta}^\eta, \\ (\rho, \eta) \overset{*}{H}_{b\gamma}^a &= (\rho, \eta) \overset{*}{H}_{b\gamma}^{\overset{c}{a}} + O_{bd}^{ae} Y_{e\gamma}^d, \\ (\rho, \eta) \overset{*}{V}_\beta^{\alpha c} &= (\rho, \eta) \overset{*}{V}_\beta^{\overset{c}{\alpha c}} + O_{\beta\eta}^{*\alpha\varepsilon} X_\varepsilon^{\eta c}, \\ (\rho, \eta) \overset{*}{V}_b^{ac} &= (\rho, \eta) \overset{*}{V}_b^{\overset{c}{ac}} + O_{bd}^{*ae} Y_e^{dc}, \end{aligned}$$

are the components of a distinguished linear (ρ, η) -connection such that the generalized tangent bundle $\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ becomes (ρ, η) -(pseudo)metrizable.

Corollary 5.2 In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, then we obtain

$$(5.7)' \quad \begin{aligned} \rho \overset{*}{H}_{\beta\gamma}^{\alpha} &= \rho \overset{*}{H}_{\beta\gamma}^{\overset{c}{\alpha}} + O_{\gamma\eta}^{\alpha\varepsilon} X_{\varepsilon\beta}^\eta, \\ \rho \overset{*}{H}_{b\gamma}^a &= \rho \overset{*}{H}_{b\gamma}^{\overset{c}{a}} + O_{bd}^{ae} Y_{e\gamma}^d, \\ \rho \overset{*}{V}_\beta^{\alpha c} &= \rho \overset{*}{V}_\beta^{\overset{c}{\alpha c}} + O_{\beta\eta}^{*\alpha\varepsilon} X_\varepsilon^{\eta c}, \\ \rho \overset{*}{V}_b^{ac} &= \rho \overset{*}{V}_b^{\overset{c}{ac}} + O_{bd}^{*ae} Y_e^{dc}, \end{aligned}$$

In the classicale case, $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$, then we obtain

$$(5.7)'' \quad \begin{aligned} \overset{*}{H}_{jk}^i &= \overset{*}{H}_{jk}^{\overset{c}{i}} + O_{kl}^{ih} X_{hj}^l, \\ \overset{*}{H}_{bk}^a &= \overset{*}{H}_{bk}^{\overset{c}{a}} + O_{bd}^{ae} Y_{ek}^d, \\ \overset{*}{V}_j^{ic} &= \overset{*}{V}_j^{\overset{c}{ic}} + O_{jl}^{*ih} X_h^{lc}, \\ \overset{*}{V}_b^{ac} &= \overset{*}{V}_b^{\overset{c}{ac}} + O_{bd}^{*ae} Y_e^{dc}, \end{aligned}$$

Theorem 5.3 Let $(\rho, \eta) \Gamma$ be a (ρ, η) -connection for the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$. If

$$\left((\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V} \right)$$

is a distinguished linear (ρ, η) -connection for the generalized tangent bundle $\left((\rho, \eta) T \overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ and

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

is a (pseudo)metrical structure, then the real local functions:

$$\begin{aligned} (\rho, \eta) \overset{*}{H}_{\beta\gamma}^\alpha &= (\rho, \eta) \overset{*}{H}_{\beta\gamma}^\alpha + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta|_\gamma}^0, \\ (\rho, \eta) \overset{*}{H}_{b\gamma}^a &= (\rho, \eta) \overset{*}{H}_{b\gamma}^a + \frac{1}{2} \tilde{g}_{be} g^{ea} \big|_\gamma^0, \\ (\rho, \eta) \overset{*}{V}_\beta^{\alpha c} &= (\rho, \eta) \overset{*}{V}_\beta^{\alpha c} + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta}^{0c} \big|, \\ (\rho, \eta) \overset{*}{V}_b^{ac} &= (\rho, \eta) \overset{*}{V}_b^{ac} + \frac{1}{2} \tilde{g}_{be} g^{ea} \big|^{0c} \end{aligned} \quad (5.8)$$

are the components of a distinguished linear (ρ, η) -connection such that the generalized tangent bundle $\left((\rho, \eta) T \overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ becomes (ρ, η) -(pseudo)metrizable.

Corollary 5.3 In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, then we obtain

$$\begin{aligned} \rho \overset{*}{H}_{\beta\gamma}^\alpha &= \rho \overset{*}{H}_{\beta\gamma}^\alpha + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta|_\gamma}^0, \\ \rho \overset{*}{H}_{b\gamma}^a &= \rho \overset{*}{H}_{b\gamma}^a + \frac{1}{2} \tilde{g}_{be} g^{ea} \big|_\gamma^0, \\ \rho \overset{*}{V}_\beta^{\alpha c} &= \rho \overset{*}{V}_\beta^{\alpha c} + \frac{1}{2} \tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta}^{0c} \big|, \\ \rho \overset{*}{V}_b^{ac} &= \rho \overset{*}{V}_b^{ac} + \frac{1}{2} \tilde{g}_{be} g^{ea} \big|^{0c} \end{aligned} \quad (5.8)'$$

In the classical case, $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$, then we obtain

$$\begin{aligned} \overset{*}{H}_{jk}^i &= \overset{*}{H}_{jk}^i + \frac{1}{2} \tilde{g}^{ih} g_{hj|k}^0, \\ \overset{*}{H}_{bk}^a &= \overset{*}{H}_{bk}^a + \frac{1}{2} \tilde{g}_{be} g^{ea} \big|_k^0, \\ \overset{*}{V}_j^{ic} &= \overset{*}{V}_j^{ic} + \frac{1}{2} \tilde{g}^{ih} g_{hj}^{0c} \big|, \\ \overset{*}{V}_b^{ac} &= \overset{*}{V}_b^{ac} + \frac{1}{2} \tilde{g}_{be} g^{ea} \big|^{0c} \end{aligned} \quad (5.8)''$$

6 Generalized Hamilton (ρ, η) -spaces, Hamilton (ρ, η) -spaces and Cartan (ρ, η) -spaces

We consider the following diagram:

$$\begin{array}{ccc} \overset{*}{E} & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\ \overset{*}{\pi} \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

such that $(E, \pi, M) = (F, \nu, N)$ and the generalized tangent bundle

$$\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$$

is (ρ, η) -(pseudo)metrizable.

Let

$$G = h_{ab} d\tilde{z}^a \otimes d\tilde{z}^b + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

be a (pseudo)metrical structure and

$$\left((\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V} \right)$$

a distinguished linear (ρ, η) -connection such that

$$(\rho, \eta) \overset{*}{D}_X G = 0, \quad \forall X \in \Gamma \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right).$$

Definition 6.1 A smooth *Hamilton fundamental function* on the dual vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$ is a mapping $\overset{*}{E} \xrightarrow{H} \mathbb{R}$ which satisfies the following conditions:

1. $H \circ \overset{*}{u} \in C^\infty(M)$, for any $\overset{*}{u} \in \Gamma \left(\overset{*}{E}, \overset{*}{\pi}, M \right) \setminus \left\{ \overset{*}{0} \right\}$;
2. $H \circ \overset{*}{0} \in C^0(M)$, where $\overset{*}{0}$ means the null section of $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$.

If $\left(U, \overset{*}{s}_U \right)$ is a local vector $(m+r)$ -chart for $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$, then real function

$$H^{ab} \overset{put}{=} \frac{\partial^2 H}{\partial p_a \partial p_b} \overset{put}{=} \frac{\partial}{\partial p_a} \left(\frac{\partial}{\partial p_b} (H) \right)$$

is defined on $\overset{*}{\pi}^{-1}(U)$.

Definition 6.2 If for any local vector $m+r$ -chart $\left(U, \overset{*}{s}_U \right)$ of $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$, we have:

$$(6.2) \quad \text{rank} \left\| H^{ab} \left(\overset{*}{u}_x \right) \right\| = r,$$

for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U) \setminus \left\{ \overset{*}{0}_x \right\}$, then we will say that *the Hamiltonian H is regular*.

Proposition 6.1 *If the Hamiltonian H is regular, then for any local vector $m + r$ -chart (U, s_U) of (E, π, M) , we obtain the real functions \tilde{H}_{ba} locally defined by*

$$(6.3) \quad \begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tilde{H}_{ba}} & \mathbb{R} \\ u_x & \longmapsto & \tilde{H}_{ba}(u_x) \end{array},$$

where $\|\tilde{H}_{ba}(u_x)\| = \|H^{ab}(u_x)\|^{-1}$, for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$.

Definition 6.3 A smooth Cartan fundamental function on the vector bundle (E, π, M) is a smooth Lagrange fundamental function $E \xrightarrow{K} \mathbb{R}_+$ which satisfies the following conditions:

1. K is positively 1-homogenous on the fibres of vector bundle (E, π, M) ;
2. For any local vector $m + r$ -chart (U, s_U) of (E, π, M) , the hessian:

$$(6.4) \quad \|K^{2ab}(u_x)\|$$

is positively define for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$.

Definition 6.4 If the (pseudo)metrical structure G is determined by a (pseudo)metrical structure

$$g = g^{ab} d\tilde{p}_a \otimes d\tilde{p}_b \in \mathcal{T}_2^0 \left(V(\rho, \eta) T^*E, (\rho, \eta) \tau_E^*, E \right),$$

namely

$$G = \tilde{g}_{ab} d\tilde{z}^a \otimes d\tilde{z}^b + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b,$$

then the (ρ, η) -(pseudo)metrizable vector bundle

$$\left((\rho, \eta) T^*E, (\rho, \eta) \tau_E^*, E \right)$$

will be called the *generalized Hamilton (ρ, η) -space*.

In particular, if the (pseudo)metrical structure g is determined with the help of a regular Hamilton (Cartan) fundamental function, namely $g = H^{ab} d\tilde{p}_a \otimes d\tilde{p}_b$ ($g = K^{2ab} d\tilde{p}_a \otimes d\tilde{p}_b$), then the (ρ, η) -(pseudo)metrizable vector bundle

$$\left((\rho, \eta) T^*E, (\rho, \eta) \tau_E^*, E \right)$$

will be called the *Hamilton (Cartan) (ρ, η) -space*.

The generalized Hamilton (Id_{T^*M}, Id_M) -spaces, the Hamilton (Id_{T^*M}, Id_M) -spaces, and the Cartan (Id_{T^*M}, Id_M) -spaces are the usual generalized Hamilton spaces, Hamilton spaces and Cartan spaces.

Theorem 6.1 *If the (pseudo)metrical structure G is determined by a (pseudo)metrical structure*

$$g \in \mathcal{T}_2^0 \left(V(\rho, \eta) T^*E, (\rho, \eta) \tau_E^*, E \right),$$

then, the real local functions:

$$\begin{aligned}
(\rho, \eta) \overset{*}{H}_{bc}^a &= \frac{1}{2} g^{ae} \left(\Gamma \left(\overset{*}{\tilde{\rho}}, Id_E^* \right) \left(\overset{*}{\tilde{\delta}}_b \right) \tilde{g}_{ec} + \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E^* \right) \left(\overset{*}{\tilde{\delta}}_c \right) \tilde{g}_{be} - \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E^* \right) \left(\overset{*}{\tilde{\delta}}_e \right) \tilde{g}_{bc} \right. \\
(6.5) \quad &\quad \left. - \tilde{g}_{cd} L_{be}^d \circ h \circ \pi^* + \tilde{g}_{bd} L_{ec}^d \circ h \circ \pi^* - \tilde{g}_{ed} L_{bc}^d \circ h \circ \pi^* \right), \\
(\rho, \eta) \overset{*}{V}_b^{ac} &= \frac{1}{2} \tilde{g}_{be} \left(\Gamma \left(\overset{*}{\tilde{\rho}}, Id_E^* \right) \left(\overset{\cdot}{\tilde{\partial}}^c \right) g^{ea} + \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E^* \right) \left(\overset{\cdot}{\tilde{\partial}}^a \right) g^{ec} - \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E^* \right) \left(\overset{\cdot}{\tilde{\partial}}^e \right) g^{ac} \right)
\end{aligned}$$

are the components of a normal distinguished linear (ρ, η) -connection with (ρ, η) - \mathcal{H} ($\mathcal{H}\mathcal{H}$) and (ρ, η) - \mathcal{V} ($\mathcal{V}\mathcal{V}$) torsions free such that the generalized tangent bundle $\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ becomes generalized Hamilton (ρ, η) -space.

This normal distinguished linear (ρ, η) -connection will be called the *generalized linear (ρ, η) -connection of Levi-Civita type*.

Corolary 6.1 In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, then we obtain

$$\begin{aligned}
\rho \overset{*}{H}_{bc}^a &= \frac{1}{2} g^{ae} \left(\Gamma \left(\overset{*}{\tilde{\rho}}, Id_E^* \right) \left(\overset{*}{\tilde{\delta}}_b \right) \tilde{g}_{ec} + \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E^* \right) \left(\overset{*}{\tilde{\delta}}_c \right) \tilde{g}_{be} - \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E^* \right) \left(\overset{*}{\tilde{\delta}}_e \right) \tilde{g}_{bc} \right. \\
(6.5)' \quad &\quad \left. - \tilde{g}_{cd} L_{be}^d \circ \pi^* + \tilde{g}_{bd} L_{ec}^d \circ \pi^* - \tilde{g}_{ed} L_{bc}^d \circ \pi^* \right), \\
\rho \overset{*}{V}_b^{ac} &= \frac{1}{2} \tilde{g}_{be} \left(\Gamma \left(\overset{*}{\tilde{\rho}}, Id_E^* \right) \left(\overset{\cdot}{\tilde{\partial}}^c \right) g^{ea} + \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E^* \right) \left(\overset{\cdot}{\tilde{\partial}}^a \right) g^{ec} - \Gamma \left(\overset{*}{\tilde{\rho}}, Id_E^* \right) \left(\overset{\cdot}{\tilde{\partial}}^e \right) g^{ac} \right)
\end{aligned}$$

In the classicale case, $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$, then we obtain

$$\begin{aligned}
(6.5)'' \quad \overset{*}{H}_{bc}^a &= \frac{1}{2} g^{ae} \left(\overset{*}{\delta}_b \tilde{g}_{ec} + \overset{*}{\delta}_c \tilde{g}_{be} - \overset{*}{\delta}_e \tilde{g}_{bc} \right) \\
\overset{*}{V}_b^{ac} &= \frac{1}{2} \tilde{g}_{be} \left(\dot{\partial}^c g^{ea} + \dot{\partial}^a g^{ec} - \dot{\partial}^e g^{ac} \right)
\end{aligned}$$

Moreover, if $(E, \pi, M) = (TM, \tau_M, M)$, then we obtain

$$\begin{aligned}
(6.5)''' \quad \overset{*}{H}_{jk}^i &= \frac{1}{2} g^{ih} \left(\overset{*}{\delta}_j \tilde{g}_{hk} + \overset{*}{\delta}_k \tilde{g}_{jh} - \overset{*}{\delta}_h \tilde{g}_{jk} \right) \\
\overset{*}{V}_j^{ik} &= \frac{1}{2} \tilde{g}_{jh} \left(\dot{\partial}^k g^{hi} + \dot{\partial}^i g^{hk} - \dot{\partial}^h g^{ik} \right)
\end{aligned}$$

Theorem 6.2 Let $\left((\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V} \right)$ be the normal distinguished linear (ρ, η) -connection presented in the previous theorem. If

$$\overset{*}{\mathbb{T}}_{bc}^a \tilde{\delta}_a \otimes d\tilde{z}^b \otimes d\tilde{z}^c \in \mathcal{T}_{20}^{10} \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$$

and

$$\overset{*}{\mathbb{S}}_b^{ac \cdot b} \tilde{\partial} \otimes \delta \tilde{p}_a \otimes \delta \tilde{p}_c \in \mathcal{T}_{01}^{02} \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$$

such that they satisfy the conditions:

$$\mathbb{T}_{bc}^{*a} = -\mathbb{T}_{cb}^{*a}, \quad \mathbb{S}_b^{*ac} = -\mathbb{S}_b^{*ac}, \quad \forall ab, c \in \overline{1, r},$$

then the following real local functions:

$$(6.6) \quad \begin{aligned} (\rho, \eta) \tilde{H}_{bc}^{*a} &= (\rho, \eta) \tilde{H}_{bc}^{*a} + \frac{1}{2} g^{ae} \left(\tilde{g}_{ed} \mathbb{T}_{bc}^{*d} - \tilde{g}_{bd} \mathbb{T}_{ec}^{*d} + \tilde{g}_{cd} \mathbb{T}_{be}^{*d} \right), \\ (\rho, \eta) \tilde{V}_b^{*ac} &= (\rho, \eta) \tilde{V}_b^{*ac} + \frac{1}{2} \tilde{g}_{be} \cdot \left(g^{ed} \mathbb{S}_d^{*ac} - g^{ad} \mathbb{S}_d^{*ec} + g^{cd} \mathbb{S}_d^{*ae} \right) \end{aligned}$$

are the components of a normal distinguished linear (ρ, η) -connection with (ρ, η) - \mathcal{H} ($\mathcal{H}\mathcal{H}$) and (ρ, η) - \mathcal{V} ($\mathcal{V}\mathcal{V}$) torsions a priori given such that the generalized tangent bundle $\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ becomes generalized Hamilton (ρ, η) -space.

Moreover, we obtain:

$$(6.7) \quad \begin{aligned} \mathbb{T}_{bc}^{*a} &= (\rho, \eta) \tilde{H}_{bc}^{*a} - (\rho, \eta) \tilde{H}_{cb}^{*a} - L_{bc}^a \circ h \circ \pi^*, \\ \mathbb{S}_b^{*ac} &= (\rho, \eta) \tilde{V}_b^{*ac} - (\rho, \eta) \tilde{V}_b^{*ca}. \end{aligned}$$

Corollary 6.2 In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, then we obtain

$$(6.6)' \quad \begin{aligned} \rho \tilde{H}_{bc}^{*a} &= \rho \tilde{H}_{bc}^{*a} + \frac{1}{2} g^{ae} \left(\tilde{g}_{ed} \mathbb{T}_{bc}^{*d} - \tilde{g}_{bd} \mathbb{T}_{ec}^{*d} + \tilde{g}_{cd} \mathbb{T}_{be}^{*d} \right), \\ \rho \tilde{V}_b^{*ac} &= \rho \tilde{V}_b^{*ac} + \frac{1}{2} \tilde{g}_{be} \cdot \left(g^{ed} \mathbb{S}_d^{*ac} - g^{ad} \mathbb{S}_d^{*ec} + g^{cd} \mathbb{S}_d^{*ae} \right). \end{aligned}$$

and

$$(6.7)' \quad \begin{aligned} \mathbb{T}_{bc}^{*a} &= \rho \tilde{H}_{bc}^{*a} - \rho \tilde{H}_{cb}^{*a} - L_{bc}^a \circ \pi^*, \\ \mathbb{S}_b^{*ac} &= \rho \tilde{V}_b^{*ac} - \rho \tilde{V}_b^{*ca}. \end{aligned}$$

In the classical case, $(\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)$, then we obtain

$$(6.6)'' \quad \begin{aligned} \tilde{H}_{bc}^{*a} &= \tilde{H}_{bc}^{*a} + \frac{1}{2} g^{ae} \left(\tilde{g}_{ed} \mathbb{T}_{bc}^{*d} - \tilde{g}_{bd} \mathbb{T}_{ec}^{*d} + \tilde{g}_{cd} \mathbb{T}_{be}^{*d} \right), \\ \tilde{V}_b^{*ac} &= \tilde{V}_b^{*ac} + \frac{1}{2} \tilde{g}_{be} \cdot \left(g^{ed} \mathbb{S}_d^{*ac} - g^{ad} \mathbb{S}_d^{*ec} + g^{cd} \mathbb{S}_d^{*ae} \right). \end{aligned}$$

and

$$(6.7)'' \quad \begin{aligned} \mathbb{T}_{bc}^{*a} &= \tilde{H}_{bc}^{*a} - \tilde{H}_{cb}^{*a}, \\ \mathbb{S}_b^{*ac} &= \tilde{V}_b^{*ac} - \tilde{V}_b^{*ca}. \end{aligned}$$

In particular, if $(E, \pi, M) = (TM, \tau_M, M)$, then we obtain

$$(6.6)''' \quad \begin{aligned} \tilde{H}_{jk}^{*i} &= H_{jk}^{*i} + \frac{1}{2} g^{ih} \left(\tilde{g}_{hl} \mathbb{T}_{jk}^{*l} - \tilde{g}_{jl} \mathbb{T}_{hk}^{*l} + \tilde{g}_{kl} \mathbb{T}_{jh}^{*l} \right), \\ \tilde{V}_j^{*ik} &= V_j^{*ik} + \frac{1}{2} \tilde{g}_{jh} \cdot \left(g^{hl} \mathbb{S}_l^{*ik} - g^{il} \mathbb{S}_l^{*hk} + g^{kl} \mathbb{S}_l^{*ih} \right). \end{aligned}$$

and

$$(6.7)''' \quad \begin{aligned} \mathbb{T}_{jk}^{*i} &= \tilde{H}_{jk}^{*i} - \tilde{H}_{kj}^{*i}, \\ \mathbb{S}_j^{*ik} &= \tilde{V}_j^{*ik} - \tilde{V}_j^{*ki}. \end{aligned}$$

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